Spectrum of the Magnetic Schrödinger Operator in a Waveguide with Combined Boundary Conditions

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Abstract

We consider the magnetic Schrödinger operator in a two-dimensional strip. On the boundary of the strip the Dirichlet boundary condition is imposed except for a fixed segment (window), where it switches to magnetic Neumann[1]. We deal with a smooth compactly supported field as well as with the Aharonov-Bohm field. We give an estimate on the maximal length of the window, for which the discrete spectrum of the considered operator will be empty. In the case of a compactly supported field we also give a sufficient condition for the presence of eigenvalues below the essential spectrum.

1 Introduction

The existence of bound states of the Laplace operator in the strip with Dirichlet boundary conditions and Neumann "window" was proven in [2] and independently also in [5]. The so called Neumann window is represented by the segment of the length $2l$ of the boundary, on which the Dirichlet conditions are changed to Neumann. A discrete spectrum of the Laplace operator with Neumann window appears for any nonzero length of the Neumann segment. In particular, for small values of l the eigenvalue emerges from the continuous spectrum proportionally to l^4. The asymptotical estimate for small l were established in [6], while the rigorous results on asymptotical expansions were obtained in [9].

On the other hand, the results on the discrete spectrum of a magnetic Schrödinger operator in waveguide-type domains are scarce. A planar quantum waveguide with constant magnetic field and a potential well is studied in [3], where it was proved that if the potential well

[1] For the definition of magnetic Neumann boundary conditions see Section 2, Eq. (2.2)

is purely attractive, then at least one bound state will appear for any value of the magnetic field.

In this work we consider the system, where the discrete spectrum in the absence of magnetic field appears due to the perturbation of the boundary of the domain rather than due to the additional potential well. We also assume that the magnetic field is localised in the sense to be specified below. This assumption rules out the case of a constant field. As it has been recently shown in [4] the presence of a suitable magnetic field can prevent the existence of bound states in the Dirichlet strip with a sufficiently small "bump". Changing the boundary conditions to Neumann is however a stronger perturbation in the sense that the existence of a bound state in a waveguide with the bump added to a certain segment of the boundary implies the existence of a bound state in a waveguide with Neumann conditions on the same segment, see [2, Cor. 1.3]. Therefore we cannot mimick the arguments of [4] in the case of the waveguide with Neumann window and a different approach is needed.

The main technical tool used in [4] is a modified version of the Hardy inequality for the magnetic Dirichlet quadratic form in the two-dimensional strip. In the present paper we establish a similar inequality in order to prove the absence of a discrete spectrum of the magnetic Schrödinger operator in the straight strip with Neumann window. More exactly speaking, we give sufficient conditions on the magnetic field and the length of the window, under which the discrete spectrum is empty. The above mentioned version of Hardy inequality enables us to reduce the problem to the study of a one-dimensional Laplacian with a purely attractive potential well of a width $2l$ and a small but fixed positive potential, see Section 4.2 for the details. We then show that for l small enough such a system has no bound state. The main profit of our method is that it gives us an explicit estimate on the critical length of the window, depending on the magnetic field, which guarantees the absence of discrete spectrum.

It is of course natural to ask whether a sufficiently large Neumann window will lead to the existence of eigenvalues also in the presence of the magnetic field. In the case of a smooth and compactly supported field we give an answer to this question using a minimax-like argument.

The article is organised as follows. In Section 2 we define the mathematical objects that we work with and describe the problem. We also give the statements of the main results separately for the case of a compactly supported bounded magnetic field and for the Aharonov-Bohm field. In Section 3 we show that the essential spectrum of the Dirichlet Laplacian is not affected by the magnetic field, neither by the presence of Neumann window. Sufficient conditions for the absence of the discrete spectrum are proved in Section 4. Finally, the question of presence of eigenvalues is discussed in Section 5.

2 Statement of the problem and the main results

Let $x = (x_1, x_2)$ be Cartesian coordinates, Ω be the strip $\{x : 0 < x_2 < \pi\}$, and γ be the interval $\{x : |x_1| < l, x_2 = 0\}$. The rest of the boundary will be indicated by Γ, i.e. $\Gamma = \partial\Omega \setminus \overline{\gamma}$. We denote by $B = B(x)$ a real-valued magnetic field and assume that A is a magnetic vector potential associated with B, i.e. $A = A(x) = (a_1(x), a_2(x))$ and $B = \operatorname{curl} A = \partial_{x_1} a_2 - \partial_{x_2} a_1$. In what follows we will consider two main cases of magnetic fields B. The first case is a smooth compactly supported field. Hereinafter by this we denote the field B belonging to $C^1(\overline{\Omega})$ and vanishing in the neighbourhood of infinity. The second one is the Aharonov-Bohm field originated by the potential with components

$$a_1(x) = -\frac{\Phi \cdot (x_2 - p_2)}{(x_1 - p_1)^2 + (x_2 - p_2)^2}, \quad a_2(x) = \frac{\Phi \cdot (x_1 - p_1)}{(x_1 - p_1)^2 + (x_2 - p_2)^2}, \tag{2.1}$$

where Φ is a constant and $2\pi\Phi$ is the flux through the point $p = (p_1, p_2)$ which is assumed to be inside the strip Ω. We denote by M_0 the operator

$$(-\mathrm{i}\partial_{x_1} + a_1)^2 + (-\mathrm{i}\partial_{x_2} + a_2)^2$$

on the domain $\mathcal{D}(M_0)$ consisting of all functions $u \in C^\infty(\overline{\Omega})$ vanishing in a neighborhood of Γ and in a neighborhood of infinity and satisfying

the boundary condition

$$(-\mathrm{i}\partial_{x_2} + a_2)u(x) = 0 \quad \text{on } \gamma. \tag{2.2}$$

We will call it magnetic Neumann boundary condition. In the case of Aharonov-Bohm field, the functions $u \in \mathcal{D}(M_0)$ are assumed to vanish in a neighbourhood of the point p. Clearly, the operator M_0 is non-negative and symmetric in $L^2(\Omega)$ and therefore it can be extended to a self-adjoint non-negative operator by the method of Friedrich. In what follows we will denote this extension by M. The main object of our interest is the spectrum of the operator M.

In order to formulate the main results we need to introduce some auxiliary notations. By $\Omega(\alpha,\beta)$ we will indicate the subset of Ω given by $\{x \in \Omega : \alpha < x_1 < \beta\}$ and $\Omega_\pm$ will be the subsets $\{x \in \Omega : x_1 > l\}$, $\{x \in \Omega : x_1 < -l\}$, respectively. The symbol $\mathcal{B}_r(q)$ denotes a ball of radius r centered at a point q in $\mathbb{R}^2$. The flux of the field through the ball $\mathcal{B}_r(q)$ is given by

$$\Phi_q(r) = \frac{1}{2\pi}\int_{\mathcal{B}_r(q)} B(x)\,dx.$$

Below we give the summary of the main results of the article.

Theorem 2.1. *The essential spectrum of the operator M coincides with $[1,+\infty)$.*

Theorem 2.2. *Assume that the field B is smooth and compactly supported and*

(1). There exist two balls $\mathcal{B}_{R_-}(p_-) \subset \Omega_-$, $\mathcal{B}_{R_+}(p_+) \subset \Omega_+$ so that at least one of the fluxes $\Phi_{p_\pm}(r)$ is not identically zero for $r \in [0, R_\pm]$;

(2). The inequality

$$l \le \frac{1}{12}(\kappa_- + \kappa_+) \tag{2.3}$$

holds true, where

$$\kappa_\pm := \min\left\{\pi c_\pm, \frac{\pi}{4\ln 2 + \pi|p_1^\pm|}\right\}, \tag{2.4}$$

$c_\pm$ are defined in Lemma 4.1.

Then the operator M has empty discrete spectrum.

Theorem 2.3. *Assume that the field B is the Aharonov-Bohm one with the potential given by (2.1) and*

(1). The point p is (p_1, p_2), where $p_1 < -l$;

(2). The inequality

$$l < \frac{\kappa}{6} \tag{2.5}$$

holds true, where

$$\kappa := \min\left\{\pi c, \frac{\pi}{4\ln 2 + \pi|p_1|}\right\}, \tag{2.6}$$

c is defined in Lemma 4.2.

Then the operator M has empty discrete spectrum.

The next theorem provides a condition, that guarantees the existence of discrete eigenvalues in the case of a smooth and compactly supported field.

Theorem 2.4. *Let the field B be smooth and compactly supported, $\lambda = \lambda(l)$ be the lowest eigenvalue of the Laplacian $-\Delta_{\mathcal{N},\mathcal{D}}$ in the strip Ω subject to the Dirichlet condition on Γ and Neumann condition on γ. Assume that the inequality*

$$\lambda(l) + \inf_{A} \max_{\overline{\Omega}} |A(x)|^2 < 1 \tag{2.7}$$

holds, where infimum is taken over all potentials associated with the field B. Then the operator M has non-empty discrete spectrum.

Remark 2.5. *It will be shown in the proof of Theorem 2.4 that under the hypothesis of this theorem the potential A can be chosen such that $|A|$ is bounded and of compact support. This will imply that the quantity $\inf\limits_{A} \max\limits_{\overline{\Omega}} |A(x)|^2$ in (2.7) is finite.*

Throughout the article we will often make use of some notations and it is convenient to introduce them now. The spectrum of an operator T will be indicated by $\sigma(T)$ while the essential spectrum will be denoted by $\sigma_{ess}(T)$. We will employ the symbol $\mathfrak{q}_T = \mathfrak{q}_T[\cdot,\cdot]$ for the sesquilinear form associated with a self-adjoint operator T and $\mathcal{D}(\mathfrak{q}_T)$ will be the domain of the quadratic form produced by the sesquilinear form $\mathfrak{q}_T$. The Hilbert space we will work in is $L^2(\Omega)$; we preserve the notation $(\cdot,\cdot)$ and $\|\cdot\|$ for the inner product and norm in this space. In all other cases the notations of the inner product and norm in a Hilbert space H will be equipped by a subscript H.

3 Proof of Theorem 2.1

To prove the theorem we will need some auxiliary notations and statements. Let H be a Hilbert space and S be a positive definite operator in H whose domain is dense in H. By S_1 we indicate the Friedrich's extension of the operator S and by S_2 another self-adjoint positive definite extension of S. By definition, $\mathcal{D}(\mathfrak{q}_{S_2})$ is a Hilbert space endowed with the inner product and the norm originated by the quadratic form $\mathfrak{q}_{S_2}$. Since S_1 is the Friedrich's extension of S it follows that $\mathcal{D}(\mathfrak{q}_{S_1})$ is a subspace of $\mathcal{D}(\mathfrak{q}_{S_2})$. Let $\mathcal{Q}$ be the orthogonal complement $\mathcal{D}(\mathfrak{q}_{S_1})^{\perp}$ in $\mathcal{D}(\mathfrak{q}_{S_2})$ in the inner product $\mathfrak{q}_{S_2}[\cdot,\cdot]$.

The proof of the theorem is based on the following lemma proven in [8, Lemma 3.1].

Lemma 3.1. *If each bounded subset of $\mathcal{Q}$ (in the norm $\|\cdot\|_{\mathcal{D}(\mathfrak{q}_{S_2})}$) is compact in H, then the operator $T := S_2^{-1} - S_1^{-1}$ is compact in H.*

In our case $L^2(\Omega)$ plays the role of H and $S := (-\mathrm{i}\nabla + A)^2 + 1$ with $\mathcal{D}(S) := C_0^\infty(\Omega)$. The Friedrich extension S_1 of S is in fact the extension of $(-\mathrm{i}\nabla + A)^2 + 1$ subject to Dirichlet boundary condition. We know from [4] that $\sigma_{ess}(S_1) = [2, +\infty)$. We set $S_2 := M + 1$; we naturally can treat $M + 1$ as an extension of S. If we prove that $T := S_2^{-1} - S_1^{-1}$ is compact, then the essential spectra of the operators S_1 and S_2 will coincide by the Weyl theorem (see for instance [1]). We will prove the compactness of T by Lemma 3.1. First we will establish

an auxiliary lemma. By ω we indicate some bounded subdomain of Ω with infinitely differentiable boundary such that $\operatorname{dist}(\gamma, \Omega \setminus \overline{\omega}) > 0$. In the case of Aharonov-Bohm field we also assume that the point p does not belong to ω.

Lemma 3.2. *For each function $u \in \mathcal{Q}$ the inequality*

$$\|u\| \leq c\|u\|_{L^2(\omega)},$$

holds true, where the constant c is independent on u.

Proof. In the proof of the lemma we follow the ideas of the proof of Lemma 3.3 in [8]. The domains $\mathcal{D}(\mathfrak{q}_{S_1})$ and $\mathcal{D}(\mathfrak{q}_{S_2})$ are completions of $C_0^\infty(\Omega)$ and $\mathcal{D}(M_0)$, respectively, in norm

$$\|(-\mathrm{i}\nabla + A) \cdot \|^2 + \| \cdot \|^2.$$

In the case of compactly supported field we can choose the vector potential A being from $C^1(\overline{\Omega})$ which will make this potential bounded on $\overline{\omega}$. In the case of Aharonov-Bohm field the potential is in $C^1(\overline{\omega})$ as well since the point p does not belong to ω by assumption. Therefore, each element v of $\mathcal{D}(S_2)$ belongs to $H^1(\omega)$ due to the inequality:

$$\begin{aligned}
\|v\|^2_{H^1(\omega)} &= \|(-\mathrm{i}\nabla + A)v - Av\|^2_{L^2(\omega)} + \|v\|^2_{L^2(\omega)} \\
&\leq 2\left(\|(-\mathrm{i}\nabla + A)v\|^2_{L^2(\omega)} + \|Av\|^2_{L^2(\omega)}\right) + \|v\|^2_{L^2(\omega)} \\
&\leq c\left(\|(-\mathrm{i}\nabla + A)v\|^2_{L^2(\omega)} + \|v\|^2_{L^2(\omega)}\right) = c(S_2 v, v),
\end{aligned} \tag{3.1}$$

where the constant c is independent on v.

We denote by $\chi = \chi(x)$ an infinitely differentiable function taking values from $[0, 1]$ and being equal to one in some neighbourhood of γ, which is a subdomain of ω, and vanishing outside ω. Since $S_2 \geq 1$ it follows that

$$\|S_2^{-1}u\| \leq \|u\|. \tag{3.2}$$

Let $u \in \mathcal{Q}$. Clearly, $(1 - \chi)S_2^{-1}u \in \mathcal{D}(\mathfrak{q}_{S_1}) \cap \mathcal{D}(S_2)$, thus

$$\left(S_2(1 - \chi)S_2^{-1}u, u\right) = \left((1 - \chi)S_2^{-1}u, u\right)_{\mathcal{D}(\mathfrak{q}_{S_2})} = 0.$$

Using this equality we deduce

$$\|u\|^2 = (u,u) - \big(S_2(1-\chi)S_2^{-1}u, u\big) = (S_2\chi S_2^{-1}u, u). \tag{3.3}$$

Since

$$S_2\chi S_2^{-1}u = \chi u - 2\left(\nabla(S_2^{-1}u), \nabla\chi\right)_{\mathbb{R}^2} - (S_2^{-1}u)\Delta\chi - 2\,\mathrm{i}\,(A, \nabla\chi)_{\mathbb{R}^2} S_2^{-1}u$$

due to (3.1)–(3.3) we have

$$\begin{aligned}\|u\|^2 &\le \int_\Omega \chi|u|^2\,dx + c\|u\|_{L^2(\omega)}\|S_2^{-1}u\|_{H^1(\omega)}\\ &\le c\|u\|_{L^2(\omega)}\left(\|u\| + \sqrt{(S_2^{-1}u, u)}\right) \le c\|u\|_{L^2(\omega)}\|u\|,\end{aligned}$$

where c is independent on u. This proves the lemma. $\square$

Let us finish the proof of the Theorem. Given a subset K of $\mathcal{Q}$ bounded in the norm $\|\cdot\|_{\mathcal{D}(\mathfrak{q}_{S_1})}$, we conclude that it is also bounded in $H^1(\omega)$ due to (3.1). By the well known theorem on compact embedding of $H^1(\omega)$ in $L^2(\omega)$ for each bounded domain with smooth boundary (see, for instance, [10]) we have that the set K is compact in $L^2(\omega)$. Applying now Lemma 3.2, we conclude that K is compact in $L^2(\Omega)$. Hence, the assumption of Lemma 3.1 is satisfied and operator T introduced above is compact. The proof of Theorem 2.1 is complete.

4 Absence of the discrete spectrum

This section is devoted to the proof of Theorems 2.2 and 2.3. By Theorem 2.1 we know that the essential spectrum of the operator M is $[1,+\infty)$. Thus, the equivalent formulation of the absence of the discrete spectrum is the following inequality

$$\inf\sigma(M-1) = \inf_{\substack{\|u\|=1\\ u\in\mathcal{D}(\mathfrak{q}_M)}} \left(\|(-\mathrm{i}\nabla + A)u\|^2 - \|u\|^2\right) \ge 0. \tag{4.1}$$

It will be enough to check the infimum for a $\|\cdot\|_{\mathcal{D}(\mathfrak{q}_M)}$-dense subset of $\mathcal{D}(M)$. Hence

$$\inf\sigma(M-1) = \inf_{\substack{\|u\|=1\\ u\in\mathcal{D}(M_0)}} \left(\|(-\mathrm{i}\nabla + A)u\|^2 - \|u\|^2\right) \ge 0 \tag{4.2}$$

In order to prove this we will need some auxiliary statements which will be established in the next two subsections.

4.1 A Hardy inequality

Here we state a Hardy inequality for the quadratic form of the operator M, which will be one of the crucial tools in the proofs of Theorems 2.2 and 2.3. Let $p = (p_1, p_2) \in \Omega$ be some point and the number R be such that $\mathcal{B}_R(p) \subset \overline{\Omega}$. Given a smooth compactly supported field B, we define the function $\mu(r) := \operatorname{dist}(\Phi_p(r), \mathbb{Z})$, where we recall that $\Phi_p(r)$ is the flux of the field B through the ball $\mathcal{B}_r(p)$. We introduce the function

$$c(p, R) = \begin{cases} \dfrac{1}{16 + c_1(R)c_2(p, R)}, & \text{if } \Phi_p(r) \not\equiv 0 \text{ as } r \in [0, R], \\ 0, & \text{if } \Phi_p(r) \equiv 0 \text{ as } r \in [0, R], \end{cases} \tag{4.3}$$

where

$$\begin{aligned} c_1(R) &= \frac{64 + 4R^2}{R^4}, \\ c_2(p, R) &= \frac{2R^2 c_3(p_2)c_4(R) + 4c_4(R) + 4R^2}{c_3(p_2)\,\cos^2(|p_2 - \frac{\pi}{2}| + R)}, \\ c_3(p_2) &= \pi^2 \min\{p_2^{-2}, (\pi - p_2)^{-2}\} - 1, \\ c_4(p, R) &= \max_{[0,R]} \left| \left(\frac{\mu(r)}{r} \right)' \right|, \\ c_5(R) &= \max\left\{2\mu_0^2 + 4c_5^2 c_6 \mu_0^4, c_6\right\}, \\ c_6(R) &= 4 \max\left\{ \frac{r_0^2}{\nu_0^2}, \frac{2R^3 - 3R^2 r_0 + r_0^3}{6r_0} \right\} \end{aligned} \tag{4.4}$$

and μ_0 and r_0 are defined by

$$\mu_0 := \frac{1}{\max\limits_{[0,R]} r^{-1}\mu(r)} v = \frac{r_0}{\mu(r_0)},$$

ν_0 is a smallest positive root of the Bessel function J_0.

It was shown in [4] that the function $c(p, R)$ is well defined. Finally, let us define

$$g(x_1) = \begin{cases} 1, & \text{if } |x_1| > l, \\ \dfrac{1}{4}, & \text{if } |x_1| \le l. \end{cases} \tag{4.5}$$

Lemma 4.1. *Assume that the field B is smooth and compactly supported and the condition (1) of Theorem 2.2 is satisfied for the points $p_- = (p_1^-, p_2^-)$ and $p_+ = (p_1^+, p_2^+)$, then*

$$\int_\Omega \rho(x_1)|u|^2\, dx \le \int_\Omega \left(|(-i\nabla + A)u|^2 - g(x_1)|u|^2 \right) dx, \tag{4.6}$$

holds for all $u \in \mathcal{D}(M_0)$, where

$$\rho(x_1) = \begin{cases} \dfrac{c_-}{1 + (x_1 - p_1^-)^2}, & \textit{if } -\infty < x_1 < p_1^-, \\ 0, & \textit{if } p_1^- < x_1 < p_1^+, \\ \dfrac{c_+}{1 + (x_1 - p_1^+)^2}, & \textit{if } p_1^+ < x_1 < +\infty, \end{cases} \tag{4.7}$$

and the constants $c_\pm = c(p_\pm, R_\pm)$ are given by (4.3).

Proof. We start the proof from the estimate

$$c_- \int_{\Omega(-\infty, p_1^-)} \frac{|u|^2}{1 + (x_1 - p_1^-)^2}\, dx \le \int_{\Omega(-\infty, p_1^-)} \left(|(-\mathrm{i}\nabla + A)u|^2 - |u|^2 \right) dx, \tag{4.8}$$

which is valid for all $u \in \mathcal{D}(M_0)$. The proof of this estimate follows from the calculations of [4, Sec. 6], where the similar inequality

$$c \int_\Omega \frac{|u|^2}{1 + (x_1 - p_1^-)^2}\, dx \le \int_\Omega \left(|(-\mathrm{i}\nabla + A)u|^2 - |u|^2 \right) dx, \tag{4.9}$$

is proved for all $u \in H_0^1(\Omega)$ with some constant c. The approach employed in [4, Sec. 3] can be applied to prove the inequality (4.8). We will not reproduce all the details of this proof and just note that the

only modification needed is to replace the function φ defined in [4, Eq. (3.28)] by

$$\varphi(x) := \begin{cases} 1 & \text{if } x_1 < p_1^- - \dfrac{R}{\sqrt{2}}, \\ \dfrac{\sqrt{2}(p_1^- - x_1)}{R} & \text{if } p_1^- - \dfrac{R}{\sqrt{2}} < x_1 < p_1^-, \\ 0 & \text{elsewhere,} \end{cases} \tag{4.10}$$

In the same way the inequality

$$c_+ \int_{\Omega(p_1^+,+\infty)} \frac{|u|^2}{1+(x_1-p_1^+)^2}\, dx \le \int_{\Omega(p_1^+,+\infty)} \left(|(-\mathrm{i}\nabla + A)u|^2 - |u|^2\right)\, dx, \tag{4.11}$$

holds for all $u \in \mathcal{D}(M_0)$, where $c_+ = c(p_+, R_+)$. We will make use of the diamagnetic inequality (see [7])

$$|\nabla |u|(x)| \le |(-\mathrm{i}\nabla + A)u(x)| \tag{4.12}$$

which holds pointwise almost everywhere in Ω for each $u \in \mathcal{D}(M_0)$. In addition the trivial inequality

$$\int_0^\pi |\partial_{x_2} u|^2\, dx_2 \ge \int_0^\pi g|u|^2\, dx_2 \tag{4.13}$$

holds for each fixed x_1 and all $u \in \mathcal{D}(M_0)$. The diamagnetic inequality (4.12) and the last estimate lead us to the inequality

$$\int_{\Omega(\alpha,\beta)} \left(|(-\mathrm{i}\nabla + A)u|^2\right)\, dx \ge \int_{\Omega(\alpha,\beta)} |\nabla|u||^2\, dx \ge \int_{\Omega(\alpha,\beta)} g|u|^2\, dx,$$

which is valid for all $\alpha < \beta$. Combining now this inequality with (4.8), (4.11) we arrive at the statement of the lemma. $\square$

In the case of the Aharonov-Bohm field the similar statement is true.

Lemma 4.2. *Assume that the field is generated by Aharonov-Bohm potential given by (2.1) and that the condition (1) of the theorem 2.3 is satisfied for the point* $p = (p_1, p_2)$*. Then*

$$\int_\Omega \rho(x_1)|u|^2\,dx \le \int_\Omega \left(|(-i\nabla + A)u|^2 - g(x_1)|u|^2\right)\,dx, \tag{4.14}$$

holds for all $u \in \mathcal{D}(M_0)$*, where*

$$\rho(x_1) = \begin{cases} \dfrac{c}{1+(x_1-p_1)^2}, & -\infty < x_1 < p_1, \\ 0, & p_1 < x_1 < +\infty, \end{cases} \tag{4.15}$$

the constant $c = c(p, \Phi)$ *is given by*

$$c(p,\Phi) = \frac{R^2\mu^2 c_3(p_2)\cos^2(|p_2 - \frac{\pi}{2}| + R)}{8\big(2\mu^2R^2c_3(p_2) + (8\mu^2 + 8 + c_3(p_2))(9R^2 + 16\pi^2)\big)}, \tag{4.16}$$

$\mu := \operatorname{dist}\{\Phi, \mathbb{Z}\}$*,* $c_2(p_2)$ *is the same as in (4.4).*

The proof of this lemma is the same as the one of Lemma 4.8. It is also based on similar calculations of [4, Sec. 7.1], where the inequality (4.9) was proven for Aharonov-Bohm field. Here one also needs to replace the function ϕ in [4, Eq. (3.28)] by the function φ defined in (4.10) with $p_1^- = p_1$.

4.2 A one-dimensional model

In this section we will show that the inequality (4.2) holds true if the one-dimensional Schrödinger operator $-\frac{d^2}{dx_1^2} + V$ in $L^2(\mathbb{R})$ with certain potential V is non-negative. We will consider the case of a compactly supported field and the Aharonov-Bohm field simultaneously.

In view of Lemmas 4.1 and 4.2 we have

$$\begin{aligned}\|(-\mathrm{i}\nabla + A)u\|^2 - \|u\|^2 =& \frac{1}{2}\left(\|(-\mathrm{i}\nabla + A)u\|^2 - (g\,u,u)\right) \\ &+ \frac{1}{2}\,\|(-\mathrm{i}\nabla + A)u\|^2 + \frac{1}{2}\left((g-2)\,u,u\right) \\ \ge& \frac{1}{2}\,\|(-\mathrm{i}\nabla + A)u\|^2 + \frac{1}{2}\left((\rho + g - 2)\,u,u\right),\end{aligned}$$

where g is given by (4.5). Here ρ is determined by (4.7) in the case of a compactly supported field and by (4.15) in the case of the Aharonov-Bohm field. Thus,

$$\begin{aligned}
&\inf_{\substack{\|u\|=1\\ u\in\mathcal{D}(M_0)}} \left(\|(-\mathrm{i}\nabla + A)u\|^2 - \|u\|^2\right)\\
&\qquad \geq \frac{1}{2} \inf_{\substack{\|u\|=1\\ u\in\mathcal{D}(M_0)}} \left(\|(-\mathrm{i}\nabla + A)u\|^2 + ((\rho + g - 2)\, u, u)\right).
\end{aligned} \tag{4.17}$$

By the diamagnetic inequality (4.12) we have

$$\begin{aligned}
&\inf_{\substack{\|u\|=1\\ u\in\mathcal{D}(M_0)}} \left(\|(-\mathrm{i}\nabla + A)u\|^2 - \|u\|^2\right)\\
&\qquad \geq \frac{1}{2} \inf_{\substack{\|u\|=1\\ u\in\mathcal{D}(M_0)}} \left(\|\nabla|u|\|^2 + ((\rho + g - 2)\, u, u)\right)\\
&\qquad = \frac{1}{2} \inf_{\substack{\|u\|=1\\ u\in\mathcal{D}(M_0)}} \left(\|\nabla u\|^2 + ((\rho + g - 2)\, u, u)\right)\\
&\qquad = \frac{1}{2} \inf_{\substack{\|u\|=1\\ u\in\mathcal{D}(M_0)}} \Bigg(\int_\Omega \left(|\partial_{x_1} u|^2 + |\partial_{x_2} u|^2\right)\, dx\\
&\qquad\qquad\qquad + ((\rho + g - 2)\, u, u)\Bigg).
\end{aligned}$$

Using now (4.13) we arrive at

$$\begin{aligned}
&\inf_{\substack{\|u\|=1\\ u\in\mathcal{D}(M_0)}} \left(\|(-\mathrm{i}\nabla + A)u\|^2 - \|u\|^2\right) \geq\\
&\qquad \geq \frac{1}{2} \inf_{\substack{\|u\|=1\\ u\in\mathcal{D}(M_0)}} \left(\|\partial_{x_1} u\|^2 + (\rho\, u, u) + 2((g-1)\, u, u)\right).
\end{aligned}$$

In order to establish the inequality (4.2) it is therefore enough to show that

$$\int_0^\pi \left[\int_{\mathbb{R}} |u_{x_1}(x)|^2 + \rho(x_1)|u(x)|^2 + 2(g(x_1) - 1)|u(x)|^2\, dx_1\right] dx_2 \geq 0,$$

which is equivalent to the inequality

$$\int_{\mathbb{R}} \left(|v'|^2 + \rho|v|^2 + 2(g-1)|v|^2\right)\, dx_1 \geq 0, \tag{4.18}$$

for all $v \in C_0^\infty(\mathbb{R})$. In other words, to prove Theorems 2.2 and 2.3 it is sufficient to show that the one-dimensional Schrödinger operator

$$-\frac{d^2}{dx_1^2} + \rho + 2(g-1)$$

is non-negative in $L^2(\mathbb{R})$. The proof of this fact is the main subject of the next section.

4.3 The proofs of Theorems 2.2 and 2.3

As it has been shown in the previous section to prove the absence of the eigenvalues it is sufficient to check the inequality (4.18). Due to the definition of g it can be rewritten as

$$\int_{\mathbb{R}} |v'(t)|^2 + \rho(t)|v(t)|^2\, dt \geq \frac{3}{2}\int_{-l}^{l} |v(t)|^2\, dt. \tag{4.19}$$

Let us show that under the assumptions of Theorems 2.2, respectively 2.3 this inequality holds true. We will show it in detail for the case of compactly supported field only (i.e. for Theorem 2.2); the case of the Aharonov-Bohm field is similar.

We introduce a function

$$\phi_-(t) := \begin{cases} c_-\left(\dfrac{\pi}{2} + \arctan(t - p_1^-)\right), & t < p_1^-, \\ \dfrac{\pi c_-}{2}, & t \geq p_1^-. \end{cases} \tag{4.20}$$

We remind that c_- and p_1^- are given by (4.7). Clearly, $\phi_-'(t) = \rho(t)$ for $t < p_1^-$ and $\phi_-'(t) = 0$ if $t \geq p_1^-$. Keeping these properties in mind for each $t \in (-l, l)$ we deduce the obvious equality

$$\begin{aligned} \frac{\pi c_-}{2} v(t) = \phi_-(t)v(t) &= \int_{-\infty}^{t} \left(\phi_-(s)v(s)\right)'\, ds \\ &= \int_{-\infty}^{p_1^-} \rho(s)v(s)\, ds + \int_{-\infty}^{t} \phi_-(s)v'(s)\, ds, \end{aligned}$$

where we also employ the fact that by the assumption of Theorem 2.2 we have $p_1^- < -l$. The equality obtained, definition of ϕ_- and Cauchy-Schwarz inequality give rise to an estimate

$$\begin{aligned} \frac{\pi^2 c_-^2}{4}|v(t)|^2 &\le 2\left(\left|\int_{-\infty}^{p_1^-}\rho(s)v(s)\,ds\right|^2 + \left|\int_{-\infty}^{t}\phi_-(s)v'(s)\,ds\right|^2\right) \\ &\le 2\left(\int_{-\infty}^{p_1^-}\rho(s)\,ds\int_{-\infty}^{p_1^-}\rho(s)|v(s)|^2\,ds + \int_{-\infty}^{t}\phi_-^2(s)\,ds\int_{-\infty}^{t}|v'(s)|^2\,ds\right) \\ &\le 2\left(\frac{\pi c_-}{2}\int_{-\infty}^{p_1^-}\rho(s)|v(s)|^2\,ds + \int_{-\infty}^{t}\phi_-^2(s)\,ds\int_{-\infty}^{l}|v'(s)|^2\,ds\right). \end{aligned} \tag{4.21}$$

Since the function $\phi_-(t)$ is constant for $t > p_1^-$ it follows that

$$\begin{aligned} \int_{-\infty}^{t}\phi_-^2(s)\,ds &= \int_{-\infty}^{p_1^-}\phi_-^2(s)\,ds + \phi_-^2(p_1^-)(t-p_1^-) \\ &= c_-^2\int_{-\infty}^{0}\left(\frac{\pi}{2}+\arctan(s)\right)^2 ds + \frac{\pi^2 c_-^2}{4}(t-p_1^-) \\ &= c_-^2\pi\ln 2 + \frac{\pi^2 c_-^2}{4}(t-p_1^-). \end{aligned}$$

Substituting the last equality into (4.21) and using the expression for $\phi_-(p_1^-)$ (see (4.20)) we arrive at

$$\begin{aligned} |v(t)|^2 \le 2\Bigg(&\frac{2}{\pi c_-}\int_{-\infty}^{p_1^-}\rho(s)|v(s)|^2\,ds \\ &+ \left(\frac{4\ln 2}{\pi} + (t-p_1^-)\right)\int_{-\infty}^{l}|v'(s)|^2\,ds\Bigg). \end{aligned} \tag{4.22}$$

In the case $c_- = 0$ the fraction $\frac{1}{c_-}$ in this inequality is understood as $+\infty$, so the inequality valid for all possible values of c_-. Integration (4.22) over $(-l, l)$ and using the obvious equality

$$\int_{-\infty}^{p_1^-}\rho(s)|v(s)|^2\,ds = \int_{-\infty}^{0}\rho(s)|v(s)|^2\,ds$$

lead us to the estimate

$$\int_{-l}^{l} |v(t)|^2\, dt \leq 4l \left(\frac{2}{\pi c_-} \int_{-\infty}^{0} \rho(s)|v(s)|^2\, ds \right.$$
$$\left. + \left(\frac{4 \ln 2}{\pi} - p_1^- \right) \int_{-\infty}^{l} |v'(s)|^2\, ds \right)$$
$$\leq \frac{4l}{\kappa_-} \left(2 \int_{-\infty}^{0} \rho(s)|v(s)|^2\, ds + \int_{-\infty}^{l} |v'(s)|^2\, ds \right),$$

where κ_- is given by (2.4). We can rewrite this inequality as

$$\kappa_- \int_{-l}^{l} |v(t)|^2\, dt \leq 4l \left(2 \int_{-\infty}^{0} \rho(s)|v(s)|^2\, ds + \int_{-\infty}^{l} |v'(s)|^2\, ds \right). \tag{4.23}$$

This inequality is valid also in the case of $c_- = 0$. In the same way one can easily prove similar inequality

$$\kappa_+ \int_{-l}^{l} |v(t)|^2\, dt \leq 4l \left(2 \int_{0}^{+\infty} \rho(s)|v(s)|^2\, ds + \int_{-l}^{+\infty} |v'(s)|^2\, ds \right), \tag{4.24}$$

where κ_+ is given by (2.4). We sum the inequalities (4.23) and (4.24) to get

$$(\kappa_- + \kappa_+) \int_{-l}^{l} |v(t)|^2\, dt \leq 4l \left(2 \int_{\mathbb{R}} \rho(s)|v(s)|^2\, ds + \int_{-\infty}^{l} |v'(s)|^2\, ds \right.$$
$$\left. + \int_{-l}^{+\infty} |v'(s)|^2\, ds \right).$$

This implies that

$$\int_{-l}^{l} |v(t)|^2\, dt \leq \frac{8l}{\kappa} \left(\int_{\mathbb{R}} \rho(s)|v(s)|^2\, ds + \int_{\mathbb{R}} |v'(s)|^2\, ds \right),$$

where $\kappa = \kappa_- + \kappa_+$. An immediate consequence of the last inequality is that to satisfy (4.19) it is sufficient to set

$$l \leq \frac{\kappa}{12},$$

which coincides with the inequality (2.3). This completes the proof of Theorem 2.2.

The proof of Theorem 2.3 is similar. One just needs to use the inequality (4.23) rewritten in a slightly different way:

$$\begin{aligned}\int_{-l}^{l} |v(t)|^2\,dt &\le 4l\Bigg(\frac{2}{\pi c_-}\int_{-\infty}^{0} \rho(s)|v(s)|^2\,ds \\ &\qquad + \left(\frac{4\ln 2}{\pi} - p_1^-\right)\int_{-\infty}^{l} |v'(s)|^2\,ds\Bigg) \\ &\le \frac{4l}{\kappa}\left(\int_{-\infty}^{0} \rho(s)|v(s)|^2\,ds + \int_{-\infty}^{l} |v'(s)|^2\,ds\right),\end{aligned}$$

with κ given by (2.6). This inequality will immediately imply the estimate (4.19) if the relation (2.5) is satisfied.

5 Presence of eigenvalues

In this section we will prove Theorem 2.4. We will use the formula

$$\inf \sigma(M-1) = \inf_{\substack{\|u\|=1 \\ u\in\mathcal{D}(\mathfrak{q}_M)}} \left(\|(-\mathrm{i}\nabla + A)u\|^2 - \|u\|^2\right).$$

If we find a test function $u \in \mathcal{D}(\mathfrak{q}_M)$ such that

$$\|(-\mathrm{i}\nabla + A)u\|^2 - \|u\|^2 < 0 \tag{5.1}$$

this will prove the presence of the discrete spectrum due to Theorem 2.1. Clearly, $\mathcal{D}(\mathfrak{q}_M)$ is a subspace of $H^1(\Omega)$ consisting of functions that vanish on Γ. The eigenfunction ψ of $-\Delta_{\mathcal{N},\mathcal{D}}$ associated with the lowest eigenvalue $\lambda(l)$ belongs to $\mathcal{D}(\mathfrak{q}_M)$. We can choose this eigenfunction being real-valued and normalized in $L^2(\Omega)$. Choosing ψ as a test function we have

$$\|(-\mathrm{i}\nabla + A)\psi\|^2 = \|\nabla\psi\|^2 + \|A\psi\|^2 = \lambda(l) + \|A\psi\|^2 \le \lambda(l) + \max_{\overline{\Omega}} |A|^2. \tag{5.2}$$

Here we used the normalization condition for ψ and an obvious relation $\lambda(l) = \|\nabla\psi\|^2$. By assumption the right hand side of the last inequality is less than one, hence the theorem is proved. Since the magnetic field B determines the magnetic vector potential A up to a gauge, one naturally should choose the potential with minimal value of $\max_{\overline{\Omega}} |A|^2$; this leads us to the inequality (2.7).

In conclusion let us show that the second term on the left hand side of (2.7) is finite. It is sufficient to show that it is finite for some A. Let A be some potential associated with B. Since B is smooth and compactly supported, the potential A can be chosen in $C^1(\overline{\Omega})$. Therefore it is bounded on each bounded subset of Ω. The support of B is a compact set, so there exists number $b > 0$ such that $B = 0$ as $x \in \Omega \setminus \Omega(-b, b)$, i.e. $\partial_{x_2} a_2 - \partial_{x_1} a_1 = 0$ as $x \in \Omega \setminus \Omega(-b, b)$. Since both domains $\Omega(-\infty, -b)$ and $\Omega(b, +\infty)$ are simply connected, this immediately implies the existence of functions $h_- \in C^1(\overline{\Omega(-\infty, -b)})$, $h_+ \in C^1(\overline{\Omega(b, +\infty)})$ such that $\nabla h_- = A$ as $x \in \overline{\Omega(-\infty, -b)}$, $\nabla h_+ = A$ as $x \in \overline{\Omega(b, +\infty)}$. We introduce the function

$$h(x) = \begin{cases} h_-(x)\zeta(x_1), & x_1 < -b, \\ 0, & -b \leq x_1 \leq b, \\ h_+(x)\zeta(x_1), & x_1 > b, \end{cases} \tag{5.3}$$

where $\zeta(x_1)$ is equal to one as $|x_1| > 2b$ and vanishes as $|x_1| \leq b$. By definition $h \in C^1(\overline{\Omega})$. The gauge transformation $\widetilde{A} := A - \nabla h$ leads us to a new vector potential $\widetilde{A}$ associated with the same field B. Moreover the potential $\widetilde{A}$ is compactly supported since $\nabla h = A$ if $|x_1|$ is large enough. Since $\widetilde{A} \in C^1(\overline{\Omega})$, it follows that $\max_{\overline{\Omega}} |\widetilde{A}|^2$ is finite.

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